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From Rota–Baxter algebras to pre-Lie algebras

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Abstract

Rota–Baxter algebras were introduced to solve some analytic and combinatorial problems and have appeared in many fields in mathematics and mathematical physics. Rota–Baxter algebras provide a construction of pre-Lie algebras from associative algebras. In this paper, we give all Rota–Baxter operators of weight 1 on complex associative algebras in dimension ≤ 3 and their corresponding pre-Lie algebras.

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1. Introduction

A Rota–Baxter algebra is an associative algebra A over a field \mathbf{F} with a linear operator $R : A \rightarrow A$ satisfying the Rota–Baxter relation

$$R(x)R(y) + \lambda R(xy) = R(R(x)y + xR(y)), \quad \forall x, y \in A. \quad (1.1)$$

Here, $\lambda \in \mathbf{F}$ is a fixed element which is called the weight. Obviously that for any $\lambda \neq 0$, $R \rightarrow \lambda^{-1}R$ can reduce the Rota–Baxter operator R of weight λ to be of weight $\lambda = 1$.

Rota–Baxter relation (1.1) first occurred in the work of G Baxter in 1960 to solve an analytic problem [Bax], based on a paper written by Spitzer [Sp] in 1956. In fact, the Rota–Baxter relation (1.1) generalizes the integration-by-parts formula. Rota [R1–R4], Atkinson [At] and Cartier [Ca] contributed important results. In particular, it was Rota who realized its importance in combinatorics and other fields in mathematics [R1, R2]. Since then, it has been related to many topics in mathematics and mathematical physics. For example, Rota–Baxter algebras appeared in connection with the work of Connes and Kreimer on renormalization theory in perturbative quantum field theory [CK2, CK3] (see [FG] for more details). It is also

related to Loday's dendriform algebras [Lo, LR], as well as to Aguiar's associative analogue of the classical Yang–Baxter equation [Ag1–Ag3].

However, it is difficult to construct examples of Rota–Baxter algebras. Basically, there are two ways to construct Rota–Baxter algebras. One way is to use the free Rota–Baxter algebras which in some sense are the ‘biggest’ examples. There are a lot of references on the study of free Rota–Baxter algebras ([Ca, R1, EG2, GK1, GK2] and the references therein). The other way is to get concrete examples in low dimensions, which is the main content of this paper. Although there has already existed certain works on (finite-dimensional) Rota–Baxter algebras, e.g. [Deb, Der, Mi1, Mi2, N], to our knowledge, there has been no ‘classification’ in low dimensions yet. We will give all Rota–Baxter algebras in dimension ≤ 3 . Though our study depends on direct computation through example one by one, these examples will be regarded as a guide for further development.

An application of Rota–Baxter (associative) algebras is to get some new algebraic structures. We mainly mention two classes of algebraic structures related to Rota–Baxter algebras in this paper. One class of algebras are the Loday's dendriform algebras [Lo, LR]. Dendriform algebras are equipped with an associative product which can be written as a linear combination of nonassociative compositions. These notions are motivated by the natural link between associative algebras and Lie algebras. By the work of Aguiar, Leroux and Ebrahimi-Fard [Ag1, E1, E2, Le1, Le2] the close relation of these new types of algebras to Rota–Baxter algebras as well as Nijenhuis algebras and differential algebras was established.

The other class of algebras are the pre-Lie algebras (or have other names such as left-symmetric algebras, Vinberg algebras and so on). Pre-Lie algebras are a class of nonassociative algebras coming from the study of convex homogeneous cones, affine manifolds and deformations of associative algebras [Au, G, Ki, Me, V]. As it was pointed out in [CL], the pre-Lie algebra ‘deserves more attention than it has been given’. It has also appeared in many fields in mathematics and mathematical physics, such as complex and symplectic structures on Lie groups and Lie algebras [AS, Ch, Sh], integrable systems [SS], classical and quantum Yang–Baxter equations [Bo, ES, GS, Ku1, Ku2], Poisson brackets and infinite-dimensional Lie algebras [BN, GD, Z], vertex algebras [BK], quantum field theory [CK1] and operads [CL]. In particular, an important role has been played by pre-Lie algebras in mathematical physics, especially the work of Connes-Kreimer on pre-Lie algebra structure on Feynman diagrams by the insertion–elimination operations (see [CK4] for a detailed interpretation). The same can be said of Rota–Baxter algebras. The connection of these two roles is still not clear, which might be clarified by careful study on the relation between Rota–Baxter algebras and pre-Lie algebras, as we try to do in this paper.

Since there is no suitable (matrix) representation theory of pre-Lie algebras due to their nonassociativity, it is natural to consider how to construct them from some algebraic structures which we have known. This is the ‘realization theory’. We have already obtained some experience. For example, a commutative associative algebra (A, \cdot) and its derivation D can define a Novikov algebra $(A, *)$ (which is a pre-Lie algebra with commutative right multiplication operators) by [GD, BM1, BM2],

$$x * y = x \cdot Dy, \quad \forall x, y \in A. \quad (1.2)$$

An analogue of the above construction in the version of Lie algebras is related to the classical Yang–Baxter equation. In fact, a Lie algebra $(\mathcal{G}, [,])$ and a linear map $R : \mathcal{G} \rightarrow \mathcal{G}$ satisfying

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]), \quad \forall x, y \in \mathcal{G} \quad (1.3)$$

can define a pre-Lie algebra $(\mathcal{G}, *)$ by ([BM3, GS, Ku3, Me])

$$x * y = [R(x), y], \quad \forall x, y \in \mathcal{G}. \quad (1.4)$$

Equation (1.3) is just the operator form of classical Yang–Baxter equation on a Lie algebra which was given by Semenov-Tyan-Shanskii in [Se]. Obviously, it can also be regarded as a Rota–Baxter operator of weight zero on the Lie algebra \mathcal{G} . In fact, as it was mentioned in [EGK, EG2], the Rota–Baxter relation (1.1) on associative algebras can be naturally extended to be on Lie algebras.

It is natural to consider the construction of pre-Lie algebras from (noncommutative) associative algebras. The answer is the construction from Rota–Baxter algebras. Let (A, \cdot) be an associative algebra and R be a Rota–Baxter operator. If the weight $\lambda = 0$, then from equations (1.3) and (1.4), it is obvious that the product

$$x * y = R(x) \cdot y - y \cdot R(x), \quad \forall x, y \in A \quad (1.5)$$

defines a pre-Lie algebra. When the weight $\lambda = 1$, we can see that the product

$$x * y = R(x) \cdot y - y \cdot R(x) - x \cdot y, \quad \forall x, y \in A \quad (1.6)$$

defines a pre-Lie algebra (see corollary 2.7). In fact, there are two approaches to both equations (1.5) and (1.6). One approach is from the relation between pre-Lie algebras and the operator form of the (modified) classical Yang–Baxter equation given by Golubchik and Sokolov in [GS]. The other approach is from the relation between dendriform dialgebras and Rota–Baxter algebras and pre-Lie algebras given by Aguiar and Ebrahimi-Fard [Ag1, E1, E2]. It is also natural to consider which kind of pre-Lie algebras can be obtained from Rota–Baxter algebras.

Note that for a commutative associative algebra, the inverse of an invertible derivation is just a Rota–Baxter operator of weight zero. So we would like to point out that in the above three algebraic constructions (commutative associative algebras, Lie algebras and associative algebras) of pre-Lie algebras, the corresponding linear transformations (derivations, operators satisfying classical Yang–Baxter equation and Rota–Baxter operators) have more or less relations to Rota–Baxter operators.

We have given a detailed study of Rota–Baxter operators on pre-Lie algebras of weight zero in [LHB]. A more remarkable property is that for any such Rota–Baxter pre-Lie algebra, equation (1.5) can also define a pre-Lie algebra which is called the double of the former [LHB]. Therefore, any pre-Lie algebra with its Rota–Baxter operator (of weight zero) and its doubles can construct a close category. We would like to point out that there is another different double construction of Rota–Baxter algebras defined by Ebrahimi-Fard in [EGK], that is, for any Rota–Baxter algebra (A, R) , there is a new Rota–Baxter algebra (A_R, R) which is called the double of (A, R) in [EGK], where the product in A_R is given by

$$x *_R y = R(x)y + xR(y) - xy, \quad \forall a, b \in A. \quad (1.7)$$

Moreover, all Rota–Baxter operators of weight zero on associative algebras in dimension ≤ 3 were given in [LHB] too.

In this paper, we study the Rota–Baxter operators of weight $\lambda = 1$ on associative algebras. It is easy to see that this Rota–Baxter operator is still a Rota–Baxter operator on the induced pre-Lie algebra given by equation (1.6) [EGP]. The paper is organized as follows. In section 2, we give some fundamental results and examples on Rota–Baxter algebras and pre-Lie algebras. In section 3, we give all Rota–Baxter algebras on two-dimensional complex pre-Lie algebras, and in the associative cases, we give their corresponding pre-Lie algebras. In section 4, we give all Rota–Baxter algebras on three-dimensional complex associative algebras and their corresponding pre-Lie algebras. In section 5, we give some discussion and conclusions.

Throughout this paper, the Rota–Baxter operator is of weight $\lambda = 1$ and all algebras are of finite dimension and over the complex field \mathbf{C} , unless otherwise stated. $\langle | \rangle$ stands for an associative algebra with a basis and nonzero products at each side of ‘|’.

2. Preliminaries and some examples

Let A be an associative algebra. For any $x, y \in A$, the commutator $[x, y] = xy - yx$ defines a Lie algebra. We denote the set of all Rota–Baxter operators on A of weight $\lambda = 1$ by $\text{RB}(A)$. Then the following conclusion is obvious (cf [E1, EGP, EG1], etc).

Lemma 2.1. *Let (A, \cdot) be an associative algebra.*

(1) *A linear operator $R \in \text{RB}(A)$ if and only if $1 - R \in \text{RB}(A)$, where 1 is the identity map. In particular, $0, 1 \in \text{RB}(A)$.*

(2) *Let $(A, *)$ be an algebra given by*

$$x * y = R(x) \cdot y + x \cdot R(y) - x \cdot y, \quad \forall x, y \in A. \tag{2.1}$$

*Then $(A, *)$ is an associative algebra and R is still a Rota–Baxter operator of weight 1 on $(A, *)$.*

(3) *If $R \in \text{RB}(A)$, then $B = 1 - 2R$ satisfies*

$$[B(x), B(y)] + [x, y] = B([B(x), y] + [x, B(y)]), \quad \forall x, y \in A. \tag{2.2}$$

(4) *Let A' denote the algebra defined by a product $(x, y) \rightarrow x \circ y$ on A which satisfies $x \circ y = y \cdot x$ for any $x, y \in A$, then A' is still an associative algebra and $\text{RB}(A) = \text{RB}(A')$.*

(5) *If $R \in \text{RB}(A)$ and $R^2 = R$, then for any $\alpha \in \mathbf{F}$, $N_\alpha = (1 + \alpha)R - \alpha$ satisfies the following Nijenhuis relation [CGM, Le1, Le2]:*

$$N_\alpha(x)N_\alpha(y) + N_\alpha^2(xy) = N_\alpha(N_\alpha(x)y + xN_\alpha(y)), \quad \forall x, y \in A. \tag{2.3}$$

Remark 2.2. In [Se], equation (2.1) is called the operator form of the modified classical Yang–Baxter equation on a Lie algebra.

In general, it is not easy to obtain $\text{RB}(A)$ for an arbitrary associative algebra A . We give some examples in certain special cases as follows.

Example 2.3. Let A be a commutative associative algebra which is the direct sum of fields. That is, there is a basis $\{e_1, \dots, e_n\}$ of A satisfying $e_i e_j = \delta_{ij} e_j$. Then by Rota–Baxter relation (1.1), $R = \sum_{k=1}^n r_{ik} e_k \in \text{RB}(A)$ if and only if

$$r_{lk} r_{kl} = 0, \quad \forall l \neq k,$$

and

$$r_{ii} = 0, r_{il} = 0 \quad \text{or} \quad -1, l \neq i; \quad \text{or} \quad r_{ii} = 1, r_{il} = 0 \quad \text{or} \quad 1, l \neq i.$$

In particular, a special case was given in [E1] as (for any $1 \leq s \leq n$)

$$R(e_i) = \sum_{l=i}^s e_l, 1 \leq i \leq s; \quad R(e_{s+1}) = 0, \quad R(e_i) = - \sum_{l=s+1}^{i-1} e_l, s+2 \leq i \leq n,$$

that is,

$$\begin{aligned} r_{ii} &= 1, & r_{ij} &= 1, & r_{ji} &= 0, \\ 1 \leq i < j \leq s; & & r_{kk} &= 0, & r_{kl} &= -1, \\ r_{lk} &= 0, & s+1 \leq l < k \leq n. \end{aligned}$$

and $r_{mn} = 0$ in the other cases. We also list $\text{RB}(A)$ for $n \leq 3$ in the following two sections.

Example 2.4. Let A be an associative algebra in dimension $n \geq 2$ satisfying the condition that for any two vectors $x, y \in A$, the product $x \cdot y$ is still in the subspace spanned by x, y . From [Bai], for any fixed $n \geq 2$, there are three kinds of such (non-isomorphic) algebras. Let $\{e_1, \dots, e_n\}$ be a basis of A , then A must be isomorphic to one of the following three algebras:

- (i) $e_i e_j = 0, \quad \forall i, j = 1, \dots, n;$
- (ii) $e_1 e_i = e_i, e_j e_i = 0, \quad \forall i = 1, \dots, n, \quad j = 2, \dots, n$
- (iii) $e_i e_1 = e_i, e_i e_j = 0, \quad \forall i = 1, \dots, n, \quad j = 2, \dots, n.$

It is obvious that $RB(I) = gl(n)$ (all $n \times n$ matrices). Note that type (III) is just type (II)' given in lemma 2.1. Hence $RB(II) = RB(III)$.

Moreover, we can prove that any operator $R \in RB(II)$ if and only if $R^2 = R$. In fact, let $R(e_i) = \sum_{k=1}^n r_{ik} e_k$, then by the Rota–Baxter relation (1.1), we only need to check the following equations (other equations hold naturally):

$$R(e_1)R(e_i) + R(e_i) = R(e_1 R(e_i) + R(e_1)e_i), \quad \forall i = 1, \dots, n.$$

For any i , the left-hand side is $r_{11}R(e_i) + R(e_i)$ and the right-hand side is $R^2(e_i) + r_{11}R(e_i)$. Therefore, $R \in RB(II)$ if and only if $R^2 = R$.

Furthermore, by conclusion (5) in lemma 2.1, we know that any Rota–Baxter operator R on the pre-Lie algebra of type (II) or type (III) can induce an operator $N_\alpha = (1 + \alpha)R - \alpha$ satisfying the Nijenhuis relation (2.3) for any $\alpha \in \mathbb{C}$.

On the other hand,

Definition 2.5. Let A be a vector space over a filed \mathbb{F} with a bilinear product $(x, y) \rightarrow xy$. A is called a pre-Lie algebra if for any $x, y, z \in A$,

$$(xy)z - x(yz) = (yx)z - y(xz). \tag{2.4}$$

It is obvious that all associative algebras are pre-Lie algebras. For a pre-Lie algebra A , the commutator

$$[x, y] = xy - yx, \tag{2.5}$$

defines a Lie algebra $\mathcal{G} = \mathcal{G}(A)$, which is called the sub-adjacent Lie algebra of A .

Proposition 2.6 [GS]. Let (A, \cdot) be an associative algebra. If a linear operator $R : A \rightarrow A$ satisfies the modified Yang–Baxter equation (2.2), then the new product $*$ on A given by

$$x * y = x \cdot y + y \cdot x + [R(x), y], \quad \forall x, y \in A \tag{2.6}$$

defines a pre-Lie algebra.

By proposition 2.6 and the conclusion (3) in lemma 2.1, we can get the following conclusion.

Corollary 2.7. Let A be an associative algebra and R be a Rota–Baxter operator of weight 1. Then the product given by equation (1.6), that is,

$$x * y = R(x) \cdot y - y \cdot R(x) - x \cdot y, \quad \forall x, y \in A \tag{2.7}$$

defines a pre-Lie algebra.

Definition 2.8 [Lo]. Let A be a vector space over a filed \mathbb{F} with two bilinear products denoted by \prec and \succ . (A, \prec, \succ) is called a dendriform dialgebra if for any $x, y, z \in A$,

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y * z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ x \succ (y \succ z) &= (x * y) \succ z, \end{aligned} \tag{2.8}$$

where $x * y = x \prec y + x \succ y$.

Proposition 2.9 [Ag1, Lo]. *Let (A, \langle, \rangle) be a dendriform dialgebra. Then the product given by*

$$x * y = x \langle y + x \rangle y, \quad \forall x, y \in A, \tag{2.9}$$

defines an associative algebra [Lo] and the product given by

$$x \circ y = x \rangle y - y \langle x, \quad \forall x, y \in A, \tag{2.10}$$

*defines a pre-Lie algebra [Ag1]. $(A, *)$ and (A, \circ) have the same sub-adjacent Lie algebra.*

Therefore, corollary 2.7 (and equation (1.5) and conclusion (2) in lemma 2.1) can also be obtained from the following conclusion (by a normalization of constant if necessary).

Proposition 2.10 [Ag1, E1]. *Let (A, \cdot) be an associative algebra and R be a Rota–Baxter operator of weight λ , then there is a dendriform dialgebra (A, \langle, \rangle) defined by*

$$x \langle y = x \cdot R(y) - \lambda x \cdot y, \quad x \rangle y = R(x) \cdot y, \quad \forall x, y \in A. \tag{2.11}$$

It is obvious that for a commutative associative algebra (A, \cdot) and any $R \in \text{RB}(A)$, the pre-Lie algebra $(A, *)$ given by equation (2.7) is still (A, \cdot) itself. It is also obvious that for an associative algebra (A, \cdot) , the pre-Lie algebra $(A, *)$ given by equation (2.7) when $R = 0$ is just (A, \cdot) itself and when $R = 1$ is (A', \circ) given in lemma 2.1. Moreover, we can get a more general conclusion: let (A, \cdot) be an associative algebra and $R \in \text{RB}(A)$. Let (A', \circ) be the associative algebra given in lemma 2.1. Then the pre-Lie algebra given by equation (2.7) through (A, R) is just the one given by equation (2.7) through $(A', 1 - R)$.

Example 2.11. Let (A, \cdot) be the associative algebra of type (II) given in example 2.4. Then the pre-Lie algebra $(A, *)$ given by equation (2.7) satisfies

$$e_1 * e_1 = -e_1 - \sum_{k=2}^n r_{1k} e_k, \\ e_1 * e_j = (r_{11} - 1)e_j, e_j * e_1 = -\sum_{k=2}^n r_{jk} e_k, \quad e_j * e_l = r_{j1} e_l, \forall j, l = 2, \dots, n,$$

where $R(e_i) = \sum_{k=1}^n r_{ik} e_k$ and $R^2 = R$. It is interesting that for $n = 2, 3$, the above pre-Lie algebras are associative (see the next two sections). However, it is not easy to get their classification in higher dimensions and we have not known whether they are still associative.

Corollary 2.12 [EGP]. *Let (A, \cdot) be an associative algebra and $R \in \text{RB}(A)$, then R is still a Rota–Baxter operator of weight $\lambda = 1$ on the pre-Lie algebra $(A, *)$ given by equation (2.7).*

Example 2.13. Let A be the two-dimensional associative algebra of type (II) in example 2.4, then it is easy to see that the operator R given by $R(e_1) = e_1, R(e_2) = ae_1$ (for any $a \neq 0$) is a Rota–Baxter operator of A (also see the following section). The pre-Lie algebra obtained by equation (2.7) is given by

$$e_1 * e_1 = -e_1, \quad e_1 * e_2 = e_2 * e_1 = 0, \quad e_2 * e_2 = ae_2.$$

It is a commutative associative algebra which is isomorphic to a simple form $\langle e'_1, e'_2 | e'_1 * e'_1 = e'_1, e'_2 * e'_2 = e'_2 \rangle$ (it is just the algebra given in example 2.3 in the case $n = 2$) by a linear transformation $e'_1 \rightarrow -e_1, e'_2 \rightarrow \frac{1}{a}e_2$. Note that R is a Rota–Baxter operator of $(A, *)$ under the same basis $\{e_1, e_2\}$ and the form R does not satisfy the conditions given in example 2.3. In fact, under the new basis $\{e'_1, e'_2\}$, R corresponds to the new form R' given by $R'(e'_1) = e'_1, R'(e'_2) = -e'_1$ which is consistent with the conclusion in example 2.3. This is an example that the matrix presentations of Rota–Baxter operators depend on the choice of the bases. Moreover, there is a related discussion in section 5.

3. Rota–Baxter operators on two-dimensional associative algebras and pre-Lie algebras

Let (A, \cdot) be an associative algebra or a pre-Lie algebra and $\{e_1, e_2, \dots, e_n\}$ be a basis of A . Let R be a Rota–Baxter operator of weight 1 on A . Set

$$R(e_i) = \sum_{j=1}^n r_{ij}e_j, \quad e_i \cdot e_j = \sum_{k=i}^n C_{ij}^k e_k. \tag{3.1}$$

Then r_{ij} satisfies the following equations:

$$\sum_{k,l,m=1}^n (C_{kl}^m r_{ik} r_{jl} + C_{ij}^k r_{km} - C_{kj}^l r_{ik} r_{lm} - C_{il}^k r_{jl} r_{km}) = 0, \quad \forall i, j = 1, 2, \dots, n. \tag{3.2}$$

We know that there are two one-dimensional associative algebras $(D0) = \langle e_1 | e_1 e_1 = 0 \rangle$ and $(D1) = \langle e_1 | e_1 e_1 = e_1 \rangle$. It is easy to see that $RB(D0) = \mathbf{C}$ and $RB(D1) = \{R | R(e_1) = 0 \text{ or } R(e_1) = e_1\}$.

We have known the classification of two-dimensional complex pre-Lie algebras [Bu], which includes the classification of two-dimensional complex associative algebras. The following results can be obtained by direct computation.

Proposition 3.1. *The Rota–Baxter operators on two-dimensional commutative associative algebras are given in the following table (any parameter belongs to the complex field \mathbf{C} , unless otherwise stated).*

Associative algebra A	Rota–Baxter operators $RB(A)$
(A1) $e_1 e_1 = e_1, e_2 e_2 = e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix},$ $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
(A2) $e_2 e_2 = e_2, e_1 e_2 = e_2 e_1 = e_1$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
(A3) $e_1 e_1 = e_1$	$\begin{pmatrix} 0 & 0 \\ 0 & r_{22} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & r_{22} \end{pmatrix}$
(A4) $e_i e_j = 0$	$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$
(A5) $e_1 e_1 = e_2$	$\begin{pmatrix} r_{11} & r_{12} \\ 0 & \frac{r_{11}^2}{2r_{11}-1} \end{pmatrix}, r_{11} \neq \frac{1}{2}$

There are two non-commutative associative algebras in dimension 2 $(B1) = \langle e_1 | e_2 e_1 = e_1, e_2 e_2 = e_2 \rangle$ and $(B2) = \langle e_1 | e_1 e_2 = e_1, e_2 e_2 = e_2 \rangle$. Both of them belong to the algebras given in example 2.4 in the case $n = 2$, so any Rota–Baxter operator R satisfies $R^2 = R$. Furthermore, we can know that (since many of their corresponding pre-Lie algebras are isomorphic under a basis transformation, we give a classification of these pre-Lie algebras ‘in the sense of isomorphism’, that is, the corresponding pre-Lie algebras are isomorphic to some pre-Lie algebras with simpler presentations)

$$\begin{aligned}
 \text{RB(B1)} &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies \text{(B1)} \right. \\
 &\quad \cup \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies \text{(B2)} \\
 &\quad \cup \begin{pmatrix} 1 & 0 \\ r_{21} & 0 \end{pmatrix} \implies \text{(A2)} \\
 &\quad \cup \begin{pmatrix} 0 & 0 \\ r_{21} & 1 \end{pmatrix} \implies \text{(A3)} \\
 &\quad \left. \cup \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & 1 - r_{11} \end{pmatrix}, r_{12} \neq 0, r_{11}^2 - r_{11} + r_{12}r_{21} = 0 \implies \text{(A1)} \right\}.
 \end{aligned}$$

We also have $\text{RB(B2)} = \text{RB(B1)}$ and the corresponding pre-Lie algebras are given by the conclusion before example 2.11.

Corollary 3.2. Any two-dimensional pre-Lie algebra obtained by equation (2.7) from a Rota–Baxter (associative) algebra is associative.

Corollary 3.3. Only the non-nilpotent commutative associative algebras (they are (A1), (A2), (A3)) can be obtained from two-dimensional non-commutative associative Rota–Baxter algebras by equation (2.7).

At the end of this section, we give the following conclusion by direct computation.

Proposition 3.4. The Rota–Baxter operators on two-dimensional (nonassociative) pre-Lie algebras are given in the following table.

Pre-Lie algebra A	Rota–Baxter operators RB(A)
(B3) $e_2e_1 = -e_1, e_2e_2 = e_1 - e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
(B4) $e_2e_1 = -e_1, e_2e_2 = ke_2, k \neq -1$	$k \neq 0: \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
(B5) $e_1e_2 = le_1, e_2e_1 = (l-1)e_1, e_2e_2 = e_1 + le_2, l \neq 0$	$k = 0: \begin{pmatrix} 1 & 0 \\ 0 & r_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix} r_{21} \neq 0$ $\begin{pmatrix} 0 & 0 \\ 0 & r_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_{21} & 1 \end{pmatrix} r_{21} \neq 0$
(B6) $e_1e_1 = 2e_1, e_1e_2 = e_2, e_2e_2 = e_1$	$l = 1: \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $l \neq 1: \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{l-1} & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\frac{1}{l-1} & 1 \end{pmatrix}$
(B6) $e_1e_1 = 2e_1, e_1e_2 = e_2, e_2e_2 = e_1$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Since (B6) is the unique simple pre-Lie algebra (without any ideals besides zero and itself) in dimension 2 [Bu], we have

Corollary 3.5. There is no non-trivial Rota–Baxter operator on the two-dimensional simple pre-Lie algebra, that is, only 0, 1 are the Rota–Baxter operators.

4. Rota–Baxter operators on three-dimensional associative algebras and their corresponding pre-Lie algebras

It is easy to get the classification of three-dimensional complex associative algebras (for example, see [LHB]). Then by direct computation, we have the following results.

Proposition 4.1. *The Rota–Baxter operators on three-dimensional commutative associative algebras are given in the following table.*

Associative algebra A	Rota–Baxter operators $RB(A)$
(C1) $e_i e_j = 0$	$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$
(C2) $e_3 e_3 = e_1$	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{11} \pm \sqrt{r_{11}^2 - r_{11}} \end{pmatrix}$
(C3) $\begin{cases} e_2 e_2 = e_1 \\ e_3 e_3 = e_1 \end{cases}$	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{11} & 0 \\ r_{31} & 0 & r_{11} \end{pmatrix} r_{11} = 0, 1$ $\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22} & r_{23} \\ r_{31} & -r_{23} & r_{22} \end{pmatrix}, r_{23} \neq 0, r_{22} = r_{11} \pm \sqrt{r_{11}^2 - r_{11} - r_{23}^2}$ $\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22}^- & r_{23}^+ \\ r_{31} & r_{23}^+ & r_{22}^+ \end{pmatrix} r_{23} \neq 0, r_{22}^+ = r_{11} + \sqrt{r_{11}^2 - r_{11} - r_{23}^2}$ $r_{22}^- = r_{11} - \sqrt{r_{11}^2 - r_{11} - r_{23}^2}$ $\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22}^+ & r_{23}^- \\ r_{31} & r_{23}^- & r_{22}^- \end{pmatrix} r_{23} \neq 0, r_{22}^+ = r_{11} + \sqrt{r_{11}^2 - r_{11} - r_{23}^2}$ $r_{22}^- = r_{11} - \sqrt{r_{11}^2 - r_{11} - r_{23}^2}$
(C4) $\begin{cases} e_2 e_3 = e_3 e_2 = e_1 \\ e_3 e_3 = e_2 \end{cases}$	$\begin{pmatrix} \frac{r_{22} r_{33}}{r_{22} + r_{33} - 1} & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}, r_{33} = r_{22} \pm \sqrt{r_{22}^2 - r_{22}}$ $r_{21} = \frac{2r_{32} r_{33} (1 - r_{33})}{(1 - 2r_{33})(r_{22} + r_{33} - 1)}$
(C5) $\begin{cases} e_1 e_1 = e_1 \\ e_2 e_2 = e_2 \\ e_3 e_3 = e_3 \end{cases}$	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & 0 & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & 2r_{33} - 1 & r_{33} \end{pmatrix}$ $\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ 2r_{33} - 1 & 0 & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & 2r_{33} - 1 & r_{33} \end{pmatrix}$ $\begin{pmatrix} r_{11} & 0 & 0 \\ 2r_{22} - 1 & r_{22} & 0 \\ 0 & 0 & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} & 0 & 0 \\ 2r_{22} - 1 & r_{22} & 0 \\ 2r_{33} - 1 & 2r_{33} - 1 & r_{33} \end{pmatrix}$ $\begin{pmatrix} r_{11} & 2r_{11} - 1 & 0 \\ 0 & r_{22} & 0 \\ 0 & 0 & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} & 2r_{11} - 1 & 0 \\ 0 & r_{22} & 0 \\ 2r_{33} - 1 & 2r_{33} - 1 & r_{33} \end{pmatrix}$ $\begin{pmatrix} r_{11} & 2r_{11} - 1 & 2r_{11} - 1 \\ 0 & r_{22} & 0 \\ 0 & 0 & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} & 2r_{11} - 1 & 2r_{11} - 1 \\ 0 & r_{22} & 0 \\ 0 & 2r_{33} - 1 & r_{33} \end{pmatrix}$ $\begin{pmatrix} r_{11} & 0 & 0 \\ 2r_{22} - 1 & r_{22} & 2r_{22} - 1 \\ 0 & 0 & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} & 0 & 0 \\ 2r_{22} - 1 & r_{22} & 2r_{22} - 1 \\ 2r_{33} - 1 & 0 & r_{33} \end{pmatrix}$

Associative algebra A	Rota–Baxter operators $RB(A)$
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 2r_{22} - 1 \\ 0 & 0 & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} & 2r_{11} - 1 & 2r_{11} - 1 \\ 0 & r_{22} & 2r_{22} - 1 \\ 0 & 0 & r_{33} \end{pmatrix}$ $\begin{pmatrix} r_{11} & 0 & 2r_{11} - 1 \\ 0 & r_{22} & 0 \\ 0 & 0 & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} & 0 & 2r_{11} - 1 \\ 2r_{22} - 1 & r_{22} & 2r_{22} - 1 \\ 0 & 0 & r_{33} \end{pmatrix}$ $r_{11} = 0, 1, r_{22} = 0, 1, r_{33} = 0, 1$
(C6) $\begin{cases} e_2 e_2 = e_2 \\ e_3 e_3 = e_3 \end{cases}$	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & r_{32} & 0 \end{pmatrix}, r_{22} = 0, 1, r_{32} = 0, -1$ $\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & r_{32} & 1 \end{pmatrix}, r_{22} = 0, 1, r_{32} = 0, 1$
(C7) $\begin{cases} e_1 e_3 = e_3 e_1 = e_1 \\ e_2 e_2 = e_2 \\ e_3 e_3 = e_3 \end{cases}$	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 2r_{22} - 1 \\ 0 & 0 & r_{33} \end{pmatrix}, r_{22} = 0, 1, r_{33} = 0, 1$ $\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & r_{32} & 0 \end{pmatrix}, r_{11} = 0, 1, r_{22} = 0, 1, r_{32} = 0, -1$ $\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & r_{32} & 1 \end{pmatrix}, r_{11} = 0, 1, r_{22} = 0, 1, r_{32} = 0, 1$ $\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & r_{33} \end{pmatrix}, r_{11} = 0, 1, r_{33} = 0, 1$ $\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & r_{33} \end{pmatrix}, r_{11} = 0, 1, r_{33} = 0, 1$
(C8) $e_3 e_3 = e_3$	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, r_{33} = 0, 1$
(C9) $\begin{cases} e_1 e_3 = e_3 e_1 = e_1 \\ e_3 e_3 = e_3 \end{cases}$	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & r_{22} & 0 \\ 0 & 0 & 1 - r_{11} \end{pmatrix}, r_{12} \neq 0, r_{11} = 0, 1$ $\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, r_{11} = 0, 1, r_{33} = 0, 1$ $\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & r_{11} \end{pmatrix}, r_{11} = 0, 1, r_{21} \neq 0$
(C10) $\begin{cases} e_1 e_3 = e_3 e_1 = e_1 \\ e_2 e_3 = e_3 e_2 = e_2 \\ e_3 e_3 = e_3 \end{cases}$	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & 1 - r_{11} & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, r_{12} \neq 0, r_{33} = 0, 1, r_{11} - r_{11}^2 - r_{12} r_{21} = 0$ $\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{11} & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, r_{33} = 0, 1, r_{11} = 0, 1$ $\begin{pmatrix} 1 & 0 & 0 \\ r_{21} & 0 & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, r_{33} = 0, 1$

Associative algebra A	Rota–Baxter operators $RB(A)$
	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 1 & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, r_{33} = 0, 1$
(C11) $\begin{cases} e_1 e_1 = e_2 \\ e_3 e_3 = e_3 \end{cases}$	$\begin{pmatrix} r_{22} \pm \sqrt{r_{22}^2 - r_{22}} & r_{12} & 0 \\ 0 & r_{22} & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, r_{33} = 0, 1$
(C12) $\begin{cases} e_1 e_1 = e_2 \\ e_1 e_3 = e_3 e_1 = e_1 \\ e_2 e_3 = e_3 e_2 = e_2 \\ e_3 e_3 = e_3 \end{cases}$	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{11} & 0 \\ 0 & 0 & r_{33} \end{pmatrix}, r_{33} = 0, 1, r_{11} = 0, 1$

Proposition 4.2. *The Rota–Baxter operators on three-dimensional non-commutative associative algebras and their corresponding pre-Lie algebras given by equation (2.7) (in the sense of isomorphism) are given in the following table.*

Associative algebra A	Rota–Baxter operators $RB(A)$	Pre-Lie algebra
(T1) $\begin{cases} e_1 \cdot e_2 = \frac{1}{2}e_3 \\ e_2 \cdot e_1 = -\frac{1}{2}e_3 \end{cases}$	$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & 1 - r_{11} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}, r_{11}^2 - r_{11} + r_{12}r_{21} = 0$	(T1) (C3)
	$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ 0 & 0 & \frac{r_{11}r_{22} - r_{12}r_{21}}{r_{11} + r_{22} - 1} \end{pmatrix}, r_{11} + r_{22} - 1 \neq 0$	(T1), (T2), (T3) $_{\lambda}, \lambda \neq 0$
(T2) $e_2 \cdot e_1 = -e_3$	$\begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 1 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$	(C1)
	$\begin{pmatrix} 1 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$	(T1)
	$\begin{pmatrix} r_{11} & 0 & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & \frac{r_{11}r_{22}}{r_{11} + r_{22} - 1} \end{pmatrix}, r_{11} + r_{22} - 1 \neq 0$	(T2), (T3) $_{\lambda}, \lambda \neq 0$
	$\begin{pmatrix} 0 & 0 & r_{13} \\ r_{21} & 1 & r_{23} \\ 0 & 0 & 1 \end{pmatrix} (r_{21} \neq 0), \begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & 1 & r_{23} \\ 0 & 0 & 0 \end{pmatrix} (r_{12} \neq 0)$	(C2)
	$\begin{pmatrix} 1 & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix} (r_{21} \neq 0), \begin{pmatrix} 1 & r_{12} & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & 1 \end{pmatrix} (r_{12} \neq 0)$	(C3)
	$\begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & 0 \end{pmatrix}, r_{12} \neq 0, r_{22} \neq 1$	(T2)
	$\begin{pmatrix} 1 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & 1 \end{pmatrix}, r_{12} \neq 0, r_{22} \neq 0$	(T2), (T3) $_{\lambda}, \lambda \neq 0$
	$\begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix}, r_{11} \neq 1, r_{21} \neq 0$	(T2), (T3) $_{\lambda}, \lambda \neq 0$
	$\begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 1 & r_{23} \\ 0 & 0 & 1 \end{pmatrix}, r_{11} \neq 0, r_{21} \neq 0$	(T2)

Associative algebra A	Rota–Baxter operators $RB(A)$	Pre-Lie algebra
(T3) $_{\lambda}$ $\begin{cases} e_1 \cdot e_1 = e_3 \\ e_1 \cdot e_2 = e_3 \\ e_2 \cdot e_2 = \lambda e_3 \\ \lambda \neq 0 \end{cases}$	$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$ $r_{11}^2 + r_{11}(r_{12} - 2r_{33}) + (\lambda r_{12}^2 + r_{33} - r_{12}r_{33}) = 0;$ $r_{11}r_{21} + r_{11}r_{22} + \lambda r_{12}r_{22} + r_{33}(1 - r_{11} - \lambda r_{12} - r_{21} - r_{22}) = 0;$ $r_{21}r_{11} + r_{12}r_{21} + \lambda r_{22}r_{12} - r_{21}r_{33} - \lambda r_{12}r_{33} = 0;$ $r_{21}^2 + r_{21}(r_{22} - r_{33}) + (\lambda r_{22}^2 + \lambda r_{33} - 2\lambda r_{22}r_{33}) = 0$	(T3) $_{\lambda}, \lambda \neq 0$
(T4) $\begin{cases} e_3 \cdot e_2 = e_2 \\ e_3 \cdot e_3 = e_3 \end{cases}$	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} r_{12} \neq 0$	(T5)
	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} r_{12} \neq 0$	(N1) $\begin{cases} e_1 * e_3 = e_2 \\ e_3 * e_1 = e_2 \\ e_3 * e_2 = e_2 \\ e_3 * e_3 = e_3 \end{cases}$
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(T4)
	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & 1 & 0 \\ r_{21}r_{32} & r_{32} & 0 \end{pmatrix}$	(C9)
	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & 0 & 0 \\ -r_{21}r_{32} & r_{32} & 1 \end{pmatrix}$	(C8)
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(T5)
(T5) $\begin{cases} e_2 \cdot e_3 = e_2 \\ e_3 \cdot e_3 = e_3 \end{cases}$	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & r_{32} & 1 - r_{22} \end{pmatrix}, r_{23} \neq 0, r_{22}^2 - r_{22} + r_{23}r_{32} = 0$	(C6)
(= (T4)')	RB(T4)	The same as in (T4)
(T6) $\begin{cases} e_1 \cdot e_1 = e_1 \\ e_3 \cdot e_2 = e_2 \\ e_3 \cdot e_3 = e_3 \end{cases}$	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix}, r_{11} = 0, 1, r_{31} = 0, -1$	(T6)
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & 0 & 1 \end{pmatrix}, r_{11} = 0, 1, r_{31} = 0, 1$	(T7)
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, r_{11} = 0, 1, r_{31} = 0, -1$	(C7)
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 1 \end{pmatrix}, r_{11} = 0, 1, r_{31} = 0, 1$	(C6)
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & r_{32} & 1 - r_{22} \end{pmatrix}, r_{23} \neq 0, r_{11} = 0, 1, r_{22}^2 - r_{22} + r_{23}r_{32} = 0$	(C5)

Associative algebra A	Rota-Baxter operators $RB(A)$	Pre-Lie algebra
	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{23} & r_{22} & r_{23} \\ -r_{22} & r_{32} & 1 - r_{22} \end{pmatrix}, r_{23} \neq 0$ $r_{11} = 0, 1$ $r_{22}^2 - r_{22} + r_{23}r_{32} = 0$	(C5)
	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{23} & r_{22} & r_{23} \\ 1 - r_{22} & r_{32} & 1 - r_{22} \end{pmatrix}, r_{23} \neq 0$ $r_{11} = 0, 1$ $r_{22}^2 - r_{22} + r_{23}r_{32} = 0$	(C5)
	$\begin{pmatrix} 0 & r_{12} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(N2) $\begin{cases} e_1 * e_1 = e_1 + 2e_3 \\ e_1 * e_3 = -e_3 \\ e_3 * e_1 = -e_3 \\ e_3 * e_2 = e_2 \\ e_3 e_3 = e_3 \end{cases}$
	$\begin{pmatrix} 1 & r_{12} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(N3) $\begin{cases} e_1 * e_1 = e_1 \\ e_1 * e_3 = e_3 \\ e_3 * e_1 = e_3 \\ e_3 * e_2 = e_2 \\ e_3 e_3 = e_3 \end{cases}$
	$\begin{pmatrix} 0 & r_{12} & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(T9)
	$\begin{pmatrix} 1 & r_{12} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(N4) $\begin{cases} e_1 * e_1 = e_1 \\ e_1 * e_2 = e_2 \\ e_2 * e_1 = e_2 \\ e_1 * e_3 = e_3 \\ e_3 * e_1 = e_3 \\ e_3 * e_2 = e_2 \\ e_3 e_3 = -e_3 \end{cases}$
	$\begin{pmatrix} 0 & r_{12} & -1 \\ 0 & 1 & 0 \\ 0 & r_{12} & 0 \end{pmatrix}$	(N5) $\begin{cases} e_1 * e_1 = e_1 \\ e_1 * e_2 = e_2 \\ e_2 * e_1 = e_2 \\ e_3 * e_2 = e_2 \\ e_3 e_3 = e_3 \end{cases}$
	$\begin{pmatrix} 1 & r_{12} & 1 \\ 0 & 1 & 0 \\ 0 & -r_{12} & 0 \end{pmatrix}$	(N6) $\begin{cases} e_1 * e_1 = e_1 \\ e_1 * e_2 = e_2 \\ e_2 * e_1 = e_2 \\ e_3 * e_2 = e_2 \\ e_3 e_3 = -e_3 \end{cases}$
	$\begin{pmatrix} 0 & r_{12} & -1 \\ 0 & 0 & 0 \\ 0 & -r_{12} & 1 \end{pmatrix}$	(T5)
	$\begin{pmatrix} 1 & r_{12} & 1 \\ 0 & 0 & 0 \\ 0 & r_{12} & 1 \end{pmatrix}$	(N7) $\begin{cases} e_1 * e_1 = e_1 \\ e_3 * e_2 = e_2 \\ e_3 e_3 = -e_3 \end{cases}$
(T7) $\begin{cases} e_1 \cdot e_1 = e_1 \\ e_2 \cdot e_3 = e_2 \\ e_3 \cdot e_3 = e_3 \end{cases}$ (=(T6)')	RB(T6)	The same as in (T6)
(T8) $\begin{cases} e_1 \cdot e_3 = e_1 \\ e_3 \cdot e_1 = e_1 \\ e_3 \cdot e_2 = e_2 \\ e_3 \cdot e_3 = e_3 \end{cases}$	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_{11} = 0, 1$	(T8)

Associative algebra A	Rota–Baxter operators RB(A)	Pre-Lie algebra
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_{11} = 0, 1$	(N8) $\begin{cases} e_1 * e_3 = e_1 \\ e_3 * e_1 = e_1 \\ e_2 * e_3 = e_2 \\ e_3 e_3 = e_3 \end{cases}$
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_{32} & 0 \end{pmatrix}, r_{11} = 0, 1$	(C10)
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 1 \end{pmatrix}, r_{11} = 0, 1$	(C9)
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & r_{32} & 1 - r_{22} \end{pmatrix}, \begin{matrix} r_{23} \neq 0 \\ r_{11} = 0, 1 \\ r_{22}^2 - r_{22} + r_{23}r_{32} = 0 \end{matrix}$	(C7)
	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 1 & 0 \\ r_{21}r_{32} & r_{32} & 0 \end{pmatrix}, r_{21} \neq 0$	(C10)
	$\begin{pmatrix} 1 & 0 & 0 \\ r_{21} & 0 & 0 \\ -r_{21}r_{32} & r_{32} & 1 \end{pmatrix}, r_{21} \neq 0$	(C9)
	$\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_{12} \neq 0$	(N9) $\begin{cases} e_1 * e_3 = -e_1 + e_2 \\ e_2 * e_3 = -e_2 \\ e_3 * e_1 = -e_1 + e_2 \\ e_3 * e_3 = -e_3 \end{cases}$
	$\begin{pmatrix} 1 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_{12} \neq 0$	(T8)
(T9) $\begin{cases} e_1 \cdot e_1 = e_1 \\ e_1 \cdot e_2 = e_2 \\ e_1 \cdot e_3 = e_3 \\ e_2 \cdot e_1 = e_2 \\ e_3 \cdot e_1 = e_3 \\ e_3 \cdot e_2 = e_2 \\ e_3 \cdot e_3 = e_3 \end{cases}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(T9)
	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_{32} & 0 \end{pmatrix}$	(C7)
	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix}, r_{31} = 0, 1$	(T9)
	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, r_{31} = 0, 1$	(C7)
	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 1 \end{pmatrix}, r_{31} = 0, -1$	(C7)
	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 1 \end{pmatrix}$	(C7)
	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & 0 & 1 \end{pmatrix}, r_{31} = 0, -1$	(T9)

Associative algebra A	Rota-Baxter operators RB(A)	Pre-Lie algebra
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & r_{32} & 1-r_{22} \end{pmatrix}, r_{23} \neq 0, r_{11} = 0, 1, r_{22}^2 - r_{22} + r_{23}r_{32} = 0$	(C5)
	$\begin{pmatrix} 0 & 0 & 0 \\ -r_{23} & r_{22} & r_{23} \\ r_{22}-1 & r_{32} & 1-r_{22} \end{pmatrix}, r_{23} \neq 0, r_{22}^2 - r_{22} + r_{23}r_{32} = 0$	(C5)
	$\begin{pmatrix} 1 & 0 & 0 \\ -r_{23} & r_{22} & r_{23} \\ r_{22} & r_{32} & 1-r_{22} \end{pmatrix}, r_{23} \neq 0, r_{22}^2 - r_{22} + r_{23}r_{32} = 0$	(C5)
	$\begin{pmatrix} r_{11} & 0 & -1 \\ 0 & r_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_{22} = 0, 1, r_{11} = 0, 1$	(N5)
	$\begin{pmatrix} r_{11} & 0 & 1 \\ 0 & r_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_{22} = 0, 1, r_{11} = 0, 1$	(T7)
	$\begin{pmatrix} 2 & 0 & -1 \\ 0 & r_{22} & 0 \\ 1 & 0 & 0 \end{pmatrix}, r_{22} = 0, 1$	(N5)
	$\begin{pmatrix} -1 & 0 & 1 \\ 0 & r_{22} & 0 \\ -1 & 0 & 1 \end{pmatrix}, r_{22} = 0, 1$	(T7)
	$\begin{pmatrix} r_{11} & r_{12} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_{11} = 0, 1, r_{12} \neq 0$	(N5)
	$\begin{pmatrix} r_{11} & r_{12} & -1 \\ 0 & 1 & 0 \\ 0 & r_{12} & 0 \end{pmatrix}, r_{11} = 0, 1, r_{12} \neq 0$	(N5)
	$\begin{pmatrix} r_{11} & r_{12} & 1 \\ 0 & 0 & 0 \\ 0 & r_{12} & 1 \end{pmatrix}, r_{11} = 0, 1, r_{12} \neq 0$	(T7)
	$\begin{pmatrix} r_{11} & r_{12} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_{11} = 0, 1, r_{12} \neq 0$	(T7)
	$\begin{pmatrix} 2 & r_{12} & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, r_{12} \neq 0$	(N5)
	$\begin{pmatrix} 2 & r_{12} & -1 \\ 0 & 1 & 0 \\ 1 & r_{12} & 0 \end{pmatrix}, r_{12} \neq 0$	(N5)
	$\begin{pmatrix} -1 & r_{12} & 1 \\ 0 & 0 & 0 \\ -1 & r_{12} & 1 \end{pmatrix}, r_{12} \neq 0$	(T7)
	$\begin{pmatrix} -1 & r_{12} & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, r_{12} \neq 0$	(T7)
(T10) $\begin{cases} e_3 \cdot e_1 = e_1 \\ e_3 \cdot e_2 = e_2 \\ e_3 \cdot e_3 = e_3 \end{cases}$	$\{R R^2 = R\}$	(T6), (T8), (T9), (T10), (T11), (C10)
(T11) $\begin{cases} e_1 \cdot e_3 = e_1 \\ e_2 \cdot e_3 = e_2 \\ e_3 \cdot e_3 = e_3 \end{cases}$	$\{R R^2 = R\}$	(T6), (T8), (T9), (T10), (T11), (C10)

Associative algebra A	Rota–Baxter operators $RB(A)$	Pre-Lie algebra
(T12) $\begin{cases} e_3 \cdot e_1 = e_1 \\ e_2 \cdot e_3 = e_2 \\ e_3 \cdot e_3 = e_3 \end{cases}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(T12)
	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix}$	(N8)
	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$	(C9)
	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 1 \end{pmatrix}$	(N8)
	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_{32} & 0 \end{pmatrix}$	(T4)
	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 1 \end{pmatrix}$	(C9)
	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ r_{31} & 0 & 1 \end{pmatrix}$	(T4)
	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 1 & 0 \\ r_{21}r_{32} & r_{32} & 0 \end{pmatrix}, r_{21} \neq 0$	(N1)
	$\begin{pmatrix} 1 & 0 & 0 \\ r_{21} & 0 & 0 \\ -r_{21}r_{32} & r_{32} & 1 \end{pmatrix}, r_{21} \neq 0$	(N9)
	$\begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & r_{32} & 1 - r_{22} \end{pmatrix}, r_{23} \neq 0, r_{11} = 0, 1, r_{22}^2 - r_{22} + r_{23}r_{32} = 0$	(N10) $\begin{cases} e_1 * e_1 = e_1 \\ e_1 * e_3 = e_3 \\ e_3 * e_1 = e_3 \\ e_3 * e_2 = e_2 \\ e_3 * e_3 = -e_3 \end{cases}$
		(N6)
	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22} & r_{23} \\ \frac{r_{21}(1-r_{11}-r_{22})}{r_{23}} & r_{32} & 1 - r_{22} \end{pmatrix}, r_{23} \neq 0, r_{21} \neq 0, r_{11} = 0, 1, r_{22}^2 - r_{22} + r_{23}r_{32} = 0$	(N6), (N10)
	$\begin{pmatrix} r_{11} & 0 & r_{13} \\ 0 & r_{22} & 0 \\ r_{31} & 0 & 1 - r_{11} \end{pmatrix}, r_{13} \neq 0, r_{22} = 0, 1, r_{11}^2 - r_{11} + r_{13}r_{31} = 0$	(N6), (N10)
	$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & 0 \\ r_{31} & \frac{r_{12}(1-r_{11}-r_{22})}{r_{13}} & 1 - r_{11} \end{pmatrix}, r_{13} \neq 0, r_{12} \neq 0, r_{22} = 0, 1, r_{11}^2 - r_{11} + r_{13}r_{31} = 0$	(N6), (N10)
	$\begin{pmatrix} 1 & r_{12} & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{31}r_{12} & 0 \end{pmatrix}, r_{12} \neq 0$	(N9)
	$\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 1 & 0 \\ r_{31} & -r_{31}r_{12} & 1 \end{pmatrix}, r_{12} \neq 0$	(N1)

Corollary 4.4. *The algebras of type (N1)–(N10) are the only nonassociative pre-Lie algebras obtained from three-dimensional Rota–Baxter algebras.*

Corollary 4.5. *The sub-adjacent Lie algebras of the nonassociative pre-Lie algebras obtained from three-dimensional Rota–Baxter algebras are unique up to isomorphism:*

$$\langle e_1, e_2, e_3 | [e_2, e_3] = e_2 \rangle.$$

It is the direct sum of the two-dimensional non-Abelian Lie algebra and one-dimensional center.

Corollary 4.6. *Besides the algebras of type (C4), (C11) and (C12), the three-dimensional commutative associative algebras can be obtained from noncommutative associative Rota–Baxter algebras by equation (2.7).*

5. Discussion and conclusions

From the study in the previous sections, we give the following discussion and conclusions:

- (1) We have given all the Rota–Baxter operators of weight 1 on complex associative algebras in dimension ≤ 3 . They can help us to construct pre-Lie algebras. We would like to point out that such constructions have some constraints. For example, all the pre-Lie algebras obtained from two-dimensional Rota–Baxter algebras are associative and the sub-adjacent Lie algebras of the nonassociative pre-Lie algebras obtained from three-dimensional Rota–Baxter algebras are unique up to isomorphism.
- (2) By conclusion (3) in lemma 2.1, the Rota–Baxter operators that we obtained in this paper can help us to get the examples of operators satisfying (the operator form of) the modified classical Yang–Baxter equation in the sub-adjacent Lie algebras of these associative algebras.
- (3) It is hard and less practicable to extend our study to be in higher dimensions since the Rota–Baxter relation involves the nonlinear quadratic equations (3.2). Moreover, for a Rota–Baxter algebra A , both the set $\text{RB}(A)$ and the corresponding pre-Lie algebras obtained from A rely on the choice of a basis of A and its corresponding structural constants (see example 2.13). So it might be enough to search some interesting examples (not necessary to get the whole set $\text{RB}(A)$) in higher dimensions, even in infinite dimension [E1]. In this sense, our study can be a good guide (like examples 2.3–2.4).
- (4) The construction in corollary 2.7 cannot be extended to the nonassociative pre-Lie algebras, that is, we cannot obtain pre-Lie algebras from a nonassociative Rota–Baxter pre-Lie algebra by equation (2.7). However, if the induced pre-Lie algebra $(A, *) = (A, *_1)$ from a Rota–Baxter (associative) algebra (A, \cdot, R) is still associative, then $(A, *_1, R)$ is still a Rota–Baxter algebra which can induce a new pre-Lie algebra $(A, *_2)$ with R being a Rota–Baxter operator (it is also a double construction, see [EGK, LHB]). Therefore, we can get a series of Rota–Baxter (associative) algebras $(A, *_n, R)$ for any $n \in \mathbb{N}$ or there exists some $N \in \mathbb{N}$ such that $(A, *_n, R)$ is a Rota–Baxter associative algebra for any $n < N$ and $(A, *_N, R)$ is a nonassociative Rota–Baxter pre-Lie algebra.
- (5) We have also given the Rota–Baxter operators of weight 1 on two-dimensional complex pre-Lie algebras. It is interesting to consider certain geometric structures related to these examples and the possible application in physics.

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